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ADVECTION–DIFFUSION PROCESSES AND RESIDENCE TIMES IN SEMIENCLOSED MARINE BASINS

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SUMMARY

This paper addresses the problem of estimating the residence times in a marine basin of a passive constituent released in the sea. The dispersion process is described by an advection–diffusion model and the hydrodynamics is assumed to be known. We have performed the analysis of two different scenarios: (i) basins with unidirectional flows, in three space dimensions and under the rigid lid approximation, and (ii) basins with flows forced by the tide, under the shallow water approximation. Let the random variable τ be defined as the time spent in the basin by a particle released at a given point. The probability distribution of τ is obtained from the solution of the advection–diffusion problem and the residence time of a particle is defined as the mean value of τ.

Two different numerical approximations have been used to solve the continuous problem: the finite volume and Monte Carlo methods. For both continuous and discrete formulations it is proved that if all the particles eventually leave the basin, then the residence time has a finite value. We present here the results obtained for two study cases: a two-dimensional basin with a steady flow and a one-dimensional channel with flow induced by the tide. The results obtained by the finite volume and Monte Carlo methods are in very good agreement for both scenarios.

KEY WORDS residence times; advection–diffusion; marine basins

1. INTRODUCTION

This paper addresses the problem of estimating the residence time in a semienclosed marine basin of a passive constituent released at a given point of the basin and subsequently dispersed by sea motions; in many cases it will eventually leave the basin. It is assumed that the evolution of a passive constituent results from advection by the currents, migration (in particular due to sedimentation or buoyancy), diffusion by turbulence and in general by all small-scale motions which contribute to the agitation of the sea. Thus this dispersion process may be described by means of an advection–diffusion model. Furthermore, here we assume that the hydrodynamics is known experimentally or given by a preliminary model.

We will consider two different scenarios: (i) basins with unidirectional flows and (ii) basins with flows forced by the tide. Here flows are defined as unidirectional when the sign of the flow (negative flow = inflow, positive flow = outflow) on the fluid boundaries between the basin and the open sea is time-independent; furthermore, the global balance 'total inflow = total outflow' holds. In contrast, flows forced by the tide oscillate on the fluid boundaries of the basin; furthermore, the time average over a tide period of the flow is zero.

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The analysis of the first situation could be performed in general in three space dimensions and under very few assumptions on the air–sea interface. However, to simplify the treatment of the subject and without a real loss of generality of the problem, we will assume the rigid lid approximation. The analysis of the second situation is performed by assuming the shallow water approximation, because many problems are handled under this assumption. Moreover, the analysis of advection–diffusion problems in the shallow water approximations with time-dependent flows (in particular oscillating flows) shows interesting peculiarities.

Let the time and space history of a particle be determined by a stochastic dispersion process in a basin. We will introduce the random variable τ defined as the time spent in the basin by a particle released at a given time and at a given point of the basin. The probability distribution of τ is obtained by means of the solution of the advection–diffusion problem describing the dispersion process. The residence time of a particle is then defined as the mean value of τ. For the advection–diffusion problems considered in the following, for which all the particles eventually leave the basin, it will be shown that the residence time, as defined, has a finite value. We note that the advection–diffusion equation is the forward Kolmogorov equation associated with the stochastic process τ. Solutions of first-passage time problems can be achieved by use of the backward equation.[12]

Two different numerical approximations have been used to solve the continuous problem: the finite volume and Monte Carlo methods. It will be shown that all the results proved for the continuous problem also hold for the solutions of the discrete equations obtained by the finite volume method. Numerical simulations have been performed for basins with different spatial dimensions and types of circulation. We present here the results obtained for two study cases: a two-dimensional basin with a steady flow and a one-dimensional channel with flow induced by the tide. These cases are idealized scenarious of dispersion processes in a coastal lagoon connected with the sea by narrow channels; the circulation in the lagoon can be controlled either by a pumping system or by the tide. The results obtained by the finite volume and Monte Carlo methods are in very good agreement for both scenarios.

2. BASINS WITH UNIDIRECTIONAL FLOWS UNDER THE RIGID LID ASSUMPTION

Let Ω be a bounded open and connectes set of $\mathbb{R}^3$ representing a semienclosed marine basin (Figure 1) in the rigid lid approximation.[3] The boundary Γ or Ω is assumed to be sufficiently smooth; moreover, $\Gamma = \Gamma_0 \cup \Gamma_s \cup \Gamma_f$, where Γ_0 is the air–sea interface, Γ_s represents the solid (lateral and bottom) boundaries and Γ_f the fluid (lateral) boundaries between the basin and the open sea. In rectangular co-ordinates $\mathbf{x} = (x_1, x_2, x_3)^T$, where the x_3-axis is vertically upwards, let $x_3 = 0$ and $x_3 = -h_0(x_1, x_2)$ be

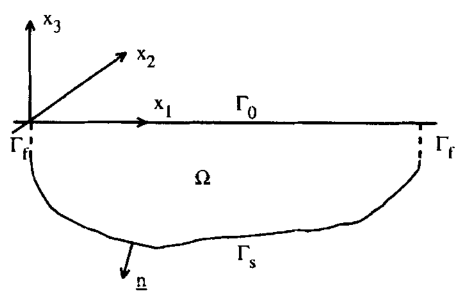

Figure 1. Representation of semienclosed basin Ω with boundary $\Gamma = \Gamma_0 \cup \Gamma_s \cup \Gamma_f.\Gamma_0$ is the air–sea interface, Γ_s represents the solid boundaries and Γ_f (broken lines) the fluid boundaries between the basin and the open sea; **n** is the outward-pointing normal on γ

the equations of the air–sea interface and the bottom respectively. The surface $x_3 = -h_0(x_1, x_2)$, $(x_1, x_2) \in \Gamma_0$, is assumed to be a single-valued function; the fluid lateral boundaries are orthogonal to Γ_0.

The velocity field in Ω is separated into an average part $\mathbf{v} = (v_1, v_2, v_3)^T$ and a fluctuating part of zero mean over a characteristic time scale t^* in the sense of the Krylov–Bogoliubov–Mitropolsky method[4]; erratic processes with characteristic times much smaller than t^* will tend to cancel each other out over a time of order t^*. The field $\mathbf{v}$, which is in general time- and space-dependent, is assumed to be divergence-free in Ω and with zero normal components to $\Gamma_0 \cup \Gamma_s$:

$$\nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega, \qquad \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_0 \cup \Gamma_s, \tag{1}$$

where $\mathbf{n}$ is the outward-pointing normal to Γ. The field $\mathbf{v}$ is assumed to be unidirectional in the sense that the sign of the flow $\mathbf{v} \cdot \mathbf{n}$ on Γ_f is assumed to be time-independent; in this case we can define the parts Γ_{f-} and Γ_{f+} of Γ_f where $\mathbf{v} \cdot \mathbf{n} \leqslant 0$ and $\mathbf{v} \cdot \mathbf{n} > 0$ respectively: $\Gamma_f = \Gamma_{f-} \cup \Gamma_{f+}$. From (1) it follows that $\mathbf{v}$ satisfies the global mass conservation equation

$$\int_{\Gamma} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}s = \int_{\Gamma_f} \mathbf{v} \cdot \mathbf{n} \, \mathrm{d}s = 0.$$

We take into account the turbulent mixing processes due to the fluctuating part of the velocity field by means of a time-independent eddy diffusivity matrix $\mathbf{K}$

$$\mathbf{K} = \begin{bmatrix} k_H & 0 & 0 \\ 0 & k_H & 0 \\ 0 & 0 & k_V \end{bmatrix},$$

where $k_H (> 0)$ and $k_V (> 0)$ are the horizontal and vertical eddy diffusivities respectively.

The migration due to the gravity and buoyancy forces affecting the dispersion of particles within the basin is introduced in the model by means of the velocity field

$$\mathbf{w} = (0, 0, w)^T \quad \text{in } \Omega,$$

where w is a constant. We assume that $w = 0$ on Γ_0 when $w > 0$ in Ω and that $w = 0$ on Γ_s when $w < 0$ in Ω. For a Stokes flow, w is expressed by

$$w = 2r^2 g(\rho_f - \rho_s)/9\nu\rho_s,$$

where r is the particle radius, g is the gravitational acceleration, ρ_s and ρ_f are the particle and sea water densities respectively and ν is the kinematic viscosity.

Let $u(t, \mathbf{x})\mathrm{d}\mathbf{x}$ be the probability of finding a particle at time t in a volume $\mathrm{d}\mathbf{x}$ around $\mathbf{x}, \mathbf{x} \in \Omega$, under the condition that at time t_0 the particle was released in the region Ω_0, a subset of Ω. Alternatively, $u(t, \mathbf{x})$ may be interpreted as the concentraiton of a passive constituent at time t and point $\mathbf{x}$ which has been released at time t_0 in Ω_0. We assume that when a particle reaches the air–sea interface Γ_0 or the solid boundaries Γ_s it is reflected inside the basin, while when it reaches the open boundaries Γ_f it generally leaves the basin, unless at Γ_f we have an advection–dominated inflow. More precisely, we define two different advection–diffusion problems depending on the behaviour of the particle on Γ_f.

Problem P1

When a particle reaches Γ_f, it leaves the basin independently of the sign of $\mathbf{v}\cdot\mathbf{n}$ on Γ_f; thus Γ_f is considered as an absorbing barrier. This situation is assumed for a diffusion–dominated flow near Γ_f. Under this assumption we are led to assume that $u(t, \mathbf{x})$ satisfies the following advection–diffusion problem:

$$\frac{\partial u}{\partial t} + \nabla\cdot[(\mathbf{v}+\mathbf{w})u - \mathbf{K}\nabla u] = 0 \quad \text{in } \Omega, \tag{2}$$

$$u(t_0, \mathbf{x}) = u_0(\mathbf{x}) > 0 \quad \text{in } \Omega_0, \qquad u(t_0, \mathbf{x}) = 0 \quad \text{in } \Omega \setminus \Omega_0, \tag{3}$$

$$(\mathbf{w}u - \mathbf{K}\nabla u)\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_0 \cup \Gamma_s, \tag{4}$$

$$u = 0 \quad \text{on } \Gamma_f. \tag{5}$$

Problem P2

When a particle reaches Γ_f, it leaves the basin when $\mathbf{v}\cdot\mathbf{n} > 0$ on Γ_f, while the total flux is assumed equal to zero when $\mathbf{v}\cdot\mathbf{n} \leqslant 0$ on Γ_f. This situation is assumed for an advection–dominated flow near Γ_f. Under this assumption we are led to assume that $u(t, \mathbf{x})$ satisfies the advection–diffusion problem defined by (2)–(4) and by the following boundary conditions on Γ_f:

$$(\mathbf{v}u - \mathbf{K}\nabla u)\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_{f-}, \qquad -\mathbf{K}\nabla u\cdot\mathbf{n} = 0 \quad \text{on } \Gamma_{f+}. \tag{6}$$

Every real situation is characterized by a specific behaviour of the particles on the open boundaries, which should be implemented in a mathematical model by suitable boundary conditions. A general form of these conditions is expressed in terms of the flux:

$$(\mathbf{v}u - \mathbf{K}\nabla u)\cdot\mathbf{n} = \Phi + v^*(u - u^*),$$

where Φ, v^* and u^* are assigned and represent the interactions between the system and the external world. The boundary conditions (5) and (6) in problems P1 and P2 respectively are special cases of the above expression for the flux, with $\Phi = 0$ and $u^* = 0$; $v^* \to \infty$ for condition (5); $v^* = 0$ on Γ_{f-} and $v^* = \mathbf{v}\cdot\mathbf{n}$ on Γ_{f+} for condition (6). Conditions (5) and (6) describe the behaviour of the particles near the open boundaries in some very representative cases, in which all the particles eventually leave the basin (Lemma 1 holds true) and the residence time (see Definitions 1 and 2) has a finite value (Theorem 1 holds true). Moreover, conditions (5) and (6) are representative of boundary conditions which are respectively independent of and dependent on the sign of $\mathbf{v}\cdot\mathbf{n}$ on Γ_f. In general, when no sources are given on Γ_f, in a first approximation the flux on Γ_f could be expressed as v^*u (i.e. with $\Phi = 0$ and $u^* = 0$ in the above expression). We would have $v^* \approx 0$ on Γ_{f-} and $v^* > 0$ on Γ_{f+}; the cases $v^* = 0$, $v^* = \mathbf{v}\cdot\mathbf{n}$ and $v^* \gg \mathbf{v}\cdot\mathbf{n}$ represent zero, advective and diffusive fluxes respectively. By assuming that $(\mathbf{v}u - \mathbf{K}\nabla u)\cdot\mathbf{n} = v^*u$ on Γ_f, a sufficient condition for the validity of Lemma 1 and Theorem 1 is given by $v^* \geqslant \mathbf{v}\cdot\mathbf{n}$. Otherwise, when sources are given on Γ_f (i.e. when Φ on Γ_{f-} and $u^* > 0$ on Γ_f), Lemma 1 does not hold. This case represents the situation of a time-continuous release.

Any solution $u(t, \mathbf{x})$ to problem P1 or P2 satisfies the maximum principle;[5,6] since $u_0(\mathbf{x}) \geqslant 0$ in Ω and $u = 0$ on Γ_f, when Dirichlet boundary conditions are assigned, it follows that $u(t, \mathbf{x}) \geqslant 0$ in Ω for $t > 0$.

Lemma 1

Let $u(t, x)$ be a solution to problem P1 or P2. Let $U_i(t)$, $i = 1, 2$ be the ith power of the $L_i(\Omega)$-norms of $u(t, \mathbf{x})$, i.e.

$$U_i(t) = \int_\Omega [u(t, \mathbf{x})]^i \,\mathrm{d}\mathbf{x},$$

and let $U_{i0} = U_i(t_0)$. Then for $i = 1,2$

$$\frac{\partial U_i}{\partial t} < 0, \tag{7}$$

$$\lim_{t\to\infty} U_i(t) = 0. \tag{8}$$

Proof. From (2)–(6) and taking into account (1), by direct calculation we obtain that

$$\frac{\partial U_1}{\partial t} + F(t) = 0, \tag{9}$$

$$\frac{1}{2}\frac{\partial U_2}{\partial t} + \int_\Omega \mathbf{K}\,\nabla u \cdot \nabla u \,\mathrm{d}\mathbf{x} + G(t) = 0, \tag{10}$$

where

$$F(t) = \int_{\Gamma_f} (\mathbf{v}u - \mathbf{K}\,\nabla u)\cdot \mathbf{n}\,\mathrm{d}s, \tag{11}$$

$$G(t) = \frac{1}{2}\int_{\Gamma_f} (u^2\mathbf{v} - \mathbf{K}\,\nabla u^2)\cdot \mathbf{n}\,\mathrm{d}s - \frac{1}{2}\int_{\Gamma_0\cup\Gamma_s} u^2\mathbf{w}\cdot\mathbf{n}\,\mathrm{d}s. \tag{12}$$

We have that $F(t) > 0$ and $G(t) > 0$ for both problems P1 and P2; therefore from (9) and (10) it follows that (7) holds. Furthermore, Friedrichs' first inequality holds, i.e.

$$\int_\Omega \mathbf{K}\,\nabla u \cdot \nabla u \,\mathrm{d}\mathbf{x} > \mu_0 U_2(t),$$

where μ_0 is a positive constant. Thus from (10) we obtain that

$$U_2(t) \leqslant U_{20}\mathrm{e}^{-2\mu_0(t-t_0)}. \tag{13}$$

Moreover,

$$U_1(t) \leqslant c\mathrm{e}^{-\mu_0(t-t_0)}, \quad \text{with } c = \left(U_{20}\int_\Omega \mathrm{d}x\right)^{1/2}. \tag{14}$$

From (13) and (14) it follows that (8) holds. □

Definition 1

Let $\Omega_0 = \{\mathbf{x}_0\}$ and let $\tau(t_0, \mathbf{x}_0)$ be the random variable defined by $\tau(t_0, \mathbf{x}_0)$ =time spent in Ω by a particle released at time t_0 in $\mathbf{x}_0$.

We introduce the notation

$$U_1(t; t_0, \mathbf{x}_0) = U_1(t)/U_{10}, \quad t > t_0,$$

for a particle released at time t_0 in $\mathbf{x}_0$; this ratio is a measure of the probability of finding a particle at time t in Ω. Thus we define

$$U_1(t; t_0, \mathbf{x}_0) = \text{Prob}\{\tau(t_0, \mathbf{x}_0) \geqslant t - t_0\} \qquad (U_1(t_0; t_0, \mathbf{x}_0) = 1), \tag{15}$$

$$P(t; t_0, \mathbf{x}_0) = 1 - U_1(t; t_0, \mathbf{x}_0) = \text{Prob}\{\tau(t_0, \mathbf{x}_0) < t - t_0\}. \tag{16}$$

The probability density function $p(t; t_0, \mathbf{x}_0)$ of $\tau(t_0, \mathbf{x}_0)$ is given by

$$p(t; t_0, \mathbf{x}_0) = \frac{\partial P}{\partial t} = -\frac{\partial U_1}{\partial t}. \tag{17}$$

From (8), which states that as time increases all the particles eventually leave the basin, we have that

$$\int_{t_0}^{\infty} p(t; t_0, \mathbf{x}_0)\mathrm{d}t = \lim_{t\to\infty} P(t; t_0, \mathbf{x}_0) = 1. \tag{18}$$

Definition 2

The residence time $T^*(t_0, \mathbf{x}_0)$ of a particle released at time t_0 in $\mathbf{x}_0$ is defined as

$$T^*(t_0, \mathbf{x}_0) = \text{mean value of the random variable } \tau(t_0, \mathbf{x}_0).$$

Theorem 1

Under the assumption of Lemma 1 for $u(t, \mathbf{x})$ let

$$T(t; t_0, \mathbf{x}_0) = \int_{t_0}^{t} (t' - t_0)p(t'; t_0, \mathbf{x}_0)\,\mathrm{d}t', \tag{19}$$

$$\hat{T}(t; t_0, \mathbf{x}_0) = T(t; t_0, \mathbf{x}_0) + (t - t_0)U_1(t; t_0, \mathbf{x}_0). \tag{20}$$

Then

$$T^*(t_0, \mathbf{x}_0) = \lim_{t\to\infty} T(t; t_0, \mathbf{x}_0) = \lim_{t\to\infty} \hat{T}(t; t_0, \mathbf{x}_0) < \infty. \tag{21}$$

Proof. From (17) and (19) we have the identity

$$T(t; t_0, \mathbf{x}_0) = -(t - t_0)U_1(t; t_0, \mathbf{x}_0) + \int_{t_0}^{t} U_1(t'; t_0, \mathbf{x}_0)\,\mathrm{d}t';$$

thus

$$T(t; t_0, \mathbf{x}_0) \leqslant \hat{T}(t; t_0, \mathbf{x}_0) = \int_{t_0}^{t} U_1(t'; t_0, \mathbf{x}_0)\,\mathrm{d}t'.$$

From (14) it follows that

$$\lim_{t\to\infty}(t - t_0)U_1(t; t_0, \mathbf{x}_0) = 0 \quad \text{and} \quad T(t; t_0, \mathbf{x}_0) \leqslant \hat{T}(t; t_0, \mathbf{x}_0) \leqslant c/\mu_0(1 - e^{-\mu_0(t-t_0)}),$$

which imply (21). □

Remarks

(i) From (9) and (17) the probability density function $p(t;\, t_0, \mathbf{x}_0)$ is expressed as

$$p(t;\, t_0, \mathbf{x}_0) = F(t),$$

where $F(t)$ is as defined in (11) and represents the total flux on Γ_f.

(ii) The estimators of the residence time defined in (19) and (20) represent

$T(t;\, t_0, \mathbf{x}_0)$ = average residence time of the particles which left the basin before time t,

$$\hat{T}(t;\, t_0, \mathbf{x}_0) = \text{average residence time at time } t.$$

(iii) Let

$$T_2(t;\, t_0, \mathbf{x}_0) = \int_{t_0}^{t} [t - t_0 - T^*(t_0, \mathbf{x}_0)]^2 p(t;\, t_0, \mathbf{x}_0)\, \mathrm{d}t.$$

Under the assumptions of Theorem 1 we have that

$$\lim_{t\to\infty} T_2(t;\, t_0, \mathbf{x}_0) = \mathrm{var}[\tau(t_0, \mathbf{x}_0)] < \infty.$$

(iv) In a pure advective process, $\tau(t_0, \mathbf{x}_0)$ is not a random variable. It assumes either a finite value $T^*(t_0, \mathbf{x}_0)$ or an infinite value depending on the type of circulation in the basin (steady or oscillating flow) and on the initial condition $(t_0, \mathbf{x}_0)$. The residence time $T^*(t_0, \mathbf{x}_0)$ may be computed from the equation of motion. In this limit case the variance is zero and the distribution $P(t;\, t_0, \mathbf{x}_0)$ and the density $p(t;\, t_0, \mathbf{x}_0)$ are represented by a step function and a Dirac δ-function respectively centred on the mean value $T^*(t_0, \mathbf{x}_0)$. The superimposition of diffusion on a pure advective process produces a smoothing in this limit distribution.

3. BASINS WITH FLOWS FORCED BY THE TIDE UNDER THE SHALLOW WATER ASSUMPTION

Let now Ω be a bounded open and connected set of $\mathbb{R}^2$ representing the projection of the bottom surface $x_3 = -h_0(\mathbf{x})$, $\mathbf{x} = (x_1, x_2)$, of the basin on the plane $x_3 = 0$. Let $\Gamma = \Gamma_s \cup \Gamma_f$, where Γ_s and Γ_f are the solid and fluid boundaries respectively, be the boundary of Ω. Let $x_3 = \eta(t, \mathbf{x})$, $\mathbf{x} \in \Omega$, be a single-valued function which defines the sea surface. The total depth $h(t, \mathbf{x})$ of the basin is given by

$$h(t, \mathbf{x}) = \eta(t, \mathbf{x}) + h_0(\mathbf{x}). \tag{22}$$

We will assume that

$$|\eta(t, \mathbf{x})| \ll h_0(\mathbf{x});$$

thus

$$0 < h_{\min} \leqslant h(t, \mathbf{x}) \leqslant h_{\max}. \tag{23}$$

In the shallow water approximation the velocity field $\mathbf{v}$ represents averaged values over the depth of the basin $(-h_0, \eta)$. It is in general time- and space-dependent, with zero normal component to Γ_s, and the divergence of $-h\mathbf{v}$ gives the time derivative of the sea level:

$$\frac{\partial \eta}{\partial t} + \nabla \cdot h\mathbf{v} = 0 \quad \text{in } \Omega, \tag{24}$$

$$\mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_s. \tag{25}$$

Furthermore, the sign and modulus of the flow $\mathbf{v} \cdot \mathbf{n}$ on Γ_f are time-dependent. In particular, when the flow is forced by the tide, we have a periodic law, of period Θ, with zero mean on Θ:

$$\int_t^{t+\Theta} h(t', \mathbf{x})\mathbf{v}(t', \mathbf{x}) \cdot \mathbf{n}\, \mathrm{d}t' = 0, \quad \mathbf{x} \in \Gamma_f. \tag{26}$$

Here the field $u(t, \mathbf{x})$ represents averaged values over the depth of the basin $(-h_0, \eta)$ of the probability of finding a particle at time t in a volume $\mathrm{d}\mathbf{x}$ around $\mathbf{x}$, $\mathbf{x} \in \Omega$. As in Section 2, we define two different advection–diffusion problems depending on the behaviour of the particle on Γ_f,

Problem P1′

When a particle reaches Γ_f, it leaves the basin independently of the sign of $\mathbf{v} \cdot \mathbf{n}$ on Γ_f. Under this assumption we are led to assume that $u(t, \mathbf{x})$ satisfies the following advection–diffusion problem:

$$\frac{\partial hu}{\partial t} + \nabla \cdot h(u\mathbf{v} - k_H \nabla u) = 0 \quad \text{in } \Omega \tag{27}$$

$$u(t_0, \mathbf{x}) = u_0(\mathbf{x}) > 0 \quad \text{in } \Omega_0 \qquad u(t_0, \mathbf{x}) = 0 \quad \text{in } \Omega \setminus \Omega_0, \tag{28}$$

$$-k_H \nabla u \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_s, \tag{29}$$

$$u = 0 \quad \text{on } \Gamma_f. \tag{30}$$

The balance equation (27) assumed in this work is a simplified form[7] of the general form derived by Nihoul.[8]

Problem P2′

When a particle reaches Γ_f, it leaves the basin when $\mathbf{v} \cdot \mathbf{n} > 0$ on Γ_f, while the total flux is assumed equal to zero when $\mathbf{v} \cdot \mathbf{n} \leqslant 0$ on Γ_f. Under this assumption we are led to assume that $u(t, \mathbf{x})$ satisfies the advection–diffusion problem defined by (27)–(29) and by the following boundary conditions on Γ_f:

$$(u\mathbf{v} - \mathbf{K}\nabla u) \cdot \mathbf{n} = 0 \quad \text{when } \mathbf{v} \cdot \mathbf{n} \leqslant 0, \qquad -\mathbf{K}\nabla u \cdot \mathbf{n} = 0 \quad \text{when } \mathbf{v} \cdot \mathbf{n} > 0. \tag{31}$$

For problem P1′ the boundary condition (30) is independent of the sign of the flow on Γ_f. In contrast, for problem P2′ the type of boundary condition (31) depends on the sign of the flow $\mathbf{v} \cdot \mathbf{n}$ on Γ_f. The problem (P2′) under the assumption of periodic flow on Γ_f is solved as follows. Let $t_j = t_0 + j\Theta/2, j = 0, 1, 2, \ldots$, and assume that on Γ_f

$$\mathbf{v} \cdot \mathbf{n} \leqslant 0 \quad \text{and} \quad (\mathbf{v}u - \mathbf{K}\nabla u) \cdot \mathbf{n} = 0 \quad \text{when } t \in (t_{2l}, t_{2l+1}),$$

$$\mathbf{v} \cdot \mathbf{n} > 0 \quad \text{and} \quad -\mathbf{K}\nabla u \cdot \mathbf{n} = 0 \quad \text{when } t \in (t_{2l+1}, t_{2l+2}),$$

for $l = 0, 1, 2, \ldots$. We solve problem P2′ in each interval (t_j, t_{j+1}) with the appropriate boundary conditions and assuming the continuity of $u(t, \mathbf{x})$ at times t_j. However, owing to this rule of changing the type of boundary condition on Γ_f, it follows that the time derivative of $u(t, \mathbf{x})$ is discontinuous at times t_j.

Lemma 2

Let $u(t, \mathbf{x})$ be a solution to problem P1′ or P2′. Let now $U_1(t)$, $i = 1, 2$, be defined as

$$U_i(t) = \int_\Omega h(t, \mathbf{x})[u(t, \mathbf{x})]^i \, \mathrm{d}\mathbf{x} \tag{32}$$

and let $U_{i0} = U_i(t_0)$. Then for $i = 1,2$

$$\frac{\partial U_i}{\partial t} \leqslant 0,$$

with $t \neq t_j$ for solutions to P2′ (because in this case for $t = t_j$ the time derivative of $U_i(t)$ is not defined), and

$$\lim_{t\to\infty} U_i(t) = 0.$$

Proof. From (27)–(31) and taking into account (22)–(26), we obtain that $U_1(t)$ and $U_2(t)$ satisfy equations (9) and (10) respectively, where $\mathbf{K} = \mathbf{I}k_\mathrm{H}$ and $F(t)$ and $G(t)$ are defined as

$$F(t) = \int_{\Gamma_f} h(\mathbf{v}u - k_\mathrm{H}\, \nabla u) \cdot \mathbf{n} \, \mathrm{d}s, \tag{33}$$

$$G(t) = \tfrac{1}{2} \int_{\Gamma_f} h(u^2 \mathbf{v} - k_\mathrm{H}\, \nabla u^2) \cdot \mathbf{n} \, \mathrm{d}s. \tag{34}$$

For problem P1′ we have that $F(t) > 0$ and $G(T) > 0$; thus $\partial U_i/\partial t$. For problem P2′ we have that

$$F(t) = 0 \quad \text{and} \quad G(t) > 0 \quad \text{for} \quad t \in (t_{2l}, t_{2l+1}),$$
$$F(t) > 0 \quad \text{and} \quad G(t) > 0 \quad \text{for} \quad t \in (t_{2l+1}, t_{2l+2});$$

thus

$$\frac{\partial U_1}{\partial t} = 0 \quad \text{and} \quad \frac{\partial U_2}{\partial t} < 0 \quad \text{for} \quad t \in (t_{2l}, t_{2l+1}),$$

$$\frac{\partial U_1}{\partial t} < 0 \quad \text{and} \quad \frac{\partial U_2}{\partial t} < 0 \quad \text{for} \quad t \in (t_{2l+1}, t_{2l+2}).$$

Again, by using Friedrichs' first inequality, it follows that $U_1(t)$ and $U_2(t)$ defined in (32) satisfy inequalities (13) and (14) respectively; thus $\lim_{t\to\infty} U_i(t) = 0$. □

From Lemma 2 it is possible to show that the estimators of the residence time defined in (19) and (20) for problems P1′ and P2′ satisfy the results of Theorem 1.

4. NUMERICAL APPROXIMATIONS

4.1. Finite volume method

From the properties of the discrete analogues of problems P1 and P2 (and also of P1′ and P2′ under the rigid lid assumption, which implies that $\nabla \cdot h\mathbf{v} = 0$) it is easy to prove the discrete analogues of Lemma 1 (Lemma 2) and Theorem 1. For problems P1′ and P2′ in the general case when the balance (24) holds, under the assumptions (22) and (23), the proof of discrete analogues of these statements is not straightforward. Thus a discrete approximation of problems P1′ and P2′ in the general shallow water formulation is presented in this subsectioni; its properties are shown and discrete analogues of Lemma 2 and Theorem 1 (see Lemma 4 and Theorem 2) are proved.

A grid is superimposed upon Ω; for simplicity we assume a regular grid of mesh spacings $\Delta x_i, i = 1, 2$. Let $x^k, k = 1, 2, \ldots, m$, be the kth grid point and let Ω_k be the rectangular region around $\mathbf{x}^k$ closed by the lines $x_1 = x_1^k \pm \Delta x_1/2$ and $x_2 = x_2^k \pm \Delta x_2/2$; let Γ_k be its boundary. We have that $\Omega = \cup\Omega_k$ and $\Omega_k \cap \Omega_l = 0$ for $k \neq l$. The discrete approximation of equation (27) is then obtained from the finite volume balances on Ω_k

$$\frac{\partial}{\partial t}\int_{\Omega_k} hu\,\mathrm{d}\mathbf{x} + \int_{\Gamma_k} h(\mathbf{v}u - k_{\mathrm{H}}\nabla u)\cdot\mathbf{n}\,\mathrm{d}s = 0 \tag{35}$$

by means of suitable approximations of the integrals in Ω_k and of the fluxes on Γ_k. We obtain the semidiscrete equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{D}(t)\mathbf{u}(t) + [\mathbf{A}(t) + \mathbf{B}(t)]\mathbf{u}(t) = 0, \qquad \mathbf{u}(t_0) = \mathbf{u}_0, \tag{36}$$

where $\mathbf{u}(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^{\mathrm{T}}$ and $\mathbf{D}(t)$, $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are $m \times m$ real matrices.

The matrix $\mathbf{D}(t)$ is a diagonal matrix with positive diagonal entries

$$D_{kk} = \int_{\Omega_k} h(t, \mathbf{x})\,\mathrm{d}\mathbf{x}.$$

The matrix $\mathbf{A}(t)$ is symmetric and is the contribution of the diffusion operator $-\nabla \cdot hk_{\mathrm{H}}\nabla$; in our model this operator is approximated by means of the usual five-point formulae (in two space dimensions; three-point formulae in one space dimension). From (23) we have that $\mathbf{A}(t)$ is positive definite for problem P1′ and positive semidefinite for problem P2′.

The matrix $\mathbf{B}(t)$ is the contribution of the advection operator $\nabla \cdot h\mathbf{v}$; it may be expressed as

$$\mathbf{B}(t) = \mathbf{B}_0(t) + \mathbf{B}_1(t) + \mathbf{B}_2(t).$$

$\mathbf{B}_0(t)$ is a diagonal matrix:

$$B_{0kk} = \int_{\Omega_k} \nabla \cdot [h(t, \mathbf{x})\mathbf{v}(t, \mathbf{x})]\,\mathrm{d}\mathbf{x}/2.$$

For a steady state flow or under the rigid lid assumption we have that $\nabla \cdot h\mathbf{v} = 0$ in Ω; thus $\mathbf{B}_0(t) = 0$. In the general case the semidiscrete analog of equation (24) holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{D}(t) + 2\mathbf{B}_0(t) = 0. \tag{37}$$

$\mathbf{B}_1(t)$ is a skew-symmetric matrix ($\mathbf{B}_1(t) = -\mathbf{B}_1^{\mathrm{T}}(t)$).$\mathbf{B}_2(t)$ is a non-negative diagonal matrix. For problem P1′, where the Dirichlet boundary condition $u = 0$ is given on Γ_{f}, $\mathbf{B}_2(t) = 0$. For problem P2′, where the total flux and the diffusive flux are assumed to be zero when $\mathbf{v} \cdot \mathbf{n} \leqslant 0$ and $\mathbf{v} \cdot \mathbf{n} > 0$ respectively, we have that the positive diagonal entries of $\mathbf{B}_2(t)$ correspond to points on Γ_{f} (if $\mathbf{x}^k \in \Gamma_{\mathrm{f}}$, then $B_{2kk}(t) = |h(t, \mathbf{x}^k)\mathbf{v}(t, \mathbf{x}^k) \cdot \mathbf{n}|/2$; otherwise $B_{2kk}(t) = 0$).

Furthermore, let

$$\mathbf{e} = [1, 1, \ldots, 1]^{\mathrm{T}}.$$

For problem P1′

$$\mathbf{A}(t)\mathbf{e} \geqslant 0, \qquad [\mathbf{A}(t) + \mathbf{B}^{\mathrm{T}}(t)]\mathbf{e} \geqslant 0; \tag{38}$$

The second inequality holds if the condition (44) on the cell Peclet numbers is satisfied. For problem P2′

$$\mathbf{A}(t)\mathbf{e} = 0 \quad \text{and} \quad \mathbf{B}^{\mathrm{T}}(t)\mathbf{e} = 0 \quad \text{when } \mathbf{v} \cdot \mathbf{n} \leqslant 0 \quad \text{on } \Gamma_{\mathrm{f}}, \tag{39}$$

$$\mathbf{A}(t)\mathbf{e} = 0 \quad \text{and} \quad \mathbf{B}^{\mathrm{T}}(t)\mathbf{e} = \mathbf{B}_2(t)\mathbf{e} \geqslant 0 \quad \text{when } \mathbf{v} \cdot \mathbf{n} > 0 \quad \text{on } \Gamma_{\mathrm{f}}. \tag{40}$$

An implicit method[9] is used to solve the system (36):

$$\mathbf{M}_j \mathbf{u}_j = \mathbf{D}_{j-1}\mathbf{u}_{j-1}, \quad j = 1, 2, \ldots, \tag{41}$$

where $\mathbf{u}_j = \mathbf{u}(t_j)$, with $t_j = t_0 + j\Delta t$ and $\mathbf{u}_0$ given,

$$\mathbf{M}_j = \mathbf{D}_j + \Delta t(\mathbf{A}_{j-1} + \mathbf{B}_{j-1}), \tag{42}$$

$$\mathbf{D}_j = \mathbf{D}(t_j), \qquad \mathbf{A}_j = \mathbf{A}(t_j), \qquad \mathbf{B}_j = \mathbf{B}(t_j), \qquad \mathbf{B}_{lj} = \mathbf{B}_l(t_j) \quad (l = 0, 1, 2). \tag{43}$$

Lemma 3

Let Pe_i^k, $i = 1, 2$, $k = 1, 2, \ldots, m$, be the cell Peclet numbers defined by

$$Pe_1^k = (\Delta x_1/2k_{\mathrm{H}})h(t, x_1^k + \Delta x_1/2, x_2^k)v_1(t, x_1^k + \Delta x_1/2, x_2^k)$$

and by a similar expression for Pe_2^k. Assume that

$$|Pe_i^k| < 1 \quad \text{for } i = 1, 2, k = 1, 2, \ldots, m. \tag{44}$$

Assume that the discrete analogue of equation (37) holds in the form

$$\mathbf{D}_j - \mathbf{D}_{j-1} + 2\Delta t\mathbf{B}_{0j-1} = 0. \tag{45}$$

Then the diagonal entries of $\mathbf{M}_j$ are positive, the non-diagonal entries are non-positive and $\mathbf{M}_j$ is an irreducible $\mathbf{M}$-matrix:

$$\det \mathbf{M}_j \neq 0, \qquad \mathbf{M}_j^{-1} > 0. \tag{46}$$

Proof. We only sketch the proof. From (42) and (45) we may write the matrix $\mathbf{M}_j$ in the form

$$\mathbf{M}_j = (\mathbf{D}_j + \mathbf{D}_{j-1})/2 + \Delta t(\mathbf{A}_{j-1} + \mathbf{B}_{1j-1} + \mathbf{B}_{2j-1}). \tag{47}$$

Then the proof of the statement follows from the properties of the matrices $\mathbf{D}_j$, $\mathbf{A}_j$, $\mathbf{B}_{1j}$ and $\mathbf{B}_{2j}$ and from the validity of condition (44). □

Remark

If discrete forms of equation (37) different from (45) hold, then we could define suitable matrices $\mathbf{M}_j$ different from (42) so that the results of Lemma 3 hold again.

Lemma 4

Under the assumptions of Lemma 3, if $\mathbf{u}_0 \geqslant 0$, then the sequence $\{\mathbf{u}_j\}$ defined in (41) is positive. Let

$$U_{1j} = \mathbf{e}^{\mathrm{T}}\mathbf{D}_j\mathbf{u}_j = \|\mathbf{D}_j\mathbf{u}_j\|_1, \tag{48}$$

$$U_{2j} = \mathbf{u}^{\mathrm{T}}\mathbf{D}_j\mathbf{u}_j = \|\mathbf{D}_j\mathbf{u}_j\|_2^2. \tag{49}$$

Then for problem P1′

$$U_{ij} < U_{ij-1}, \quad i = 1, 2, \tag{50}$$

while for problem P2′

$$U_{1j} = U_{1j-1} \quad \text{and} \quad U_{2j} < U_{2j-1} \quad \text{when } \mathbf{v}\cdot\mathbf{n} \leqslant 0 \quad \text{on } \Gamma_{\mathrm{f}}, \tag{51}$$

$$U_{ij} < U_{ij-1}, \quad i = 1, 2, \quad \text{when } \mathbf{v}\cdot\mathbf{n} > 0 \quad \text{on } \Gamma_{\mathrm{f}}. \tag{52}$$

Furthermore, for both problems P1′ and P2′

$$\lim_{j\to\infty} U_{ij} = 0, \quad i = 1, 2. \tag{53}$$

Thus the sequence $\{\mathbf{u}_j\}$ is convergent to the null vector.

Proof. Assume that $u_0 \geqslant 0$; then from (41) and (46) it follows that $\mathbf{u}_j > 0$. From (41) and (42) we obtain that

$$U_{1j} + \Delta t\mathbf{e}^{\mathrm{T}}(\mathbf{A}_{j-1} + \mathbf{B}_{j-1})\mathbf{u}_j = U_{1j-1}. \tag{54}$$

Then inequalities (50)–(52) for $i = 1$ follow from equation (54) and from the properties (38)–(40).

From (41) and (45) after some manipulations we obtain that

$$U_{2j} + 2\Delta t\mathbf{u}_j^{\mathrm{T}}(\mathbf{A}_{j-1} + \mathbf{B}_{2j-1})\mathbf{u}_j + S_j = U_{2j-1}, \tag{55}$$

where

$$S_j = (\mathbf{u}_j - \mathbf{u}_{j-1})^{\mathrm{T}}\mathbf{D}_{j-1}(\mathbf{u}_j - \mathbf{u}_{j-1}) \geqslant 0.$$

The eigenvalues of the matrices $\mathbf{D}_j^{-1}(\mathbf{A}_{j-1} + \mathbf{B}_{j-1})$ are real and non-negative.

Let

$$\nu_j = \text{min eigenvalue of } \mathbf{D}_j^{-1}(\mathbf{A}_{j-1} + \mathbf{B}_{j-1}), \quad j = 1, 2, \ldots.$$

For problem P1′ we have that $\nu_j \geqslant \nu_0 > 0$; from (39) and (40) it follows for problem P2′ that $\nu_j = 0$ when $\mathbf{v}\cdot\mathbf{n} \leqslant 0$ on Γ_{f} and that $\nu_j \geqslant \nu_0 > 0$ when $\mathbf{v}\cdot\mathbf{n} > 0$ on Γ_{f}. Here $\nu_0 = \min_{\nu_j>0} \nu_j$. Thus from (55) we obtain that

$$U_{2j} \leqslant U_{2j-1}/(1 + 2\nu_j\Delta t) \leqslant U_{20}(1 + 2\nu_2\Delta t)^{-\varepsilon j+\varepsilon_0}, \tag{56}$$

where $\varepsilon = 1$, $\varepsilon_0 = 0$ for problem P1′ and $\varepsilon = \frac{1}{2}$, $\varepsilon_0 = \Theta/2\Delta t$ for problem P2′. Moreover, since

$$U_{1j} \leqslant (\mathbf{e}^{\mathrm{T}}\mathbf{D}_j\mathbf{e}U_{2j})^{1/2}, \tag{57}$$

we have that

$$U_{1j} \leqslant c'(1 + 2\nu_0 \Delta t)^{-\varepsilon j/2}, \tag{58}$$

where

$$c' = \max_j [\mathbf{e}^{\mathrm{T}} \mathbf{D}_j \mathbf{e} U_{20} (1 + 2\nu_0 \Delta t)^{\varepsilon_0}]^{1/2}.$$

From (56) and (58) it follows that (53). □

We now define the discrete analogues of (32), (16), (17), (19) and (20); they are written as

$$U_{1j}(t_0, \mathbf{x}_0) = \mathbf{e}^{\mathrm{T}} \mathbf{D}_j \mathbf{u}_j / \mathbf{e}^{\mathrm{T}} \mathbf{D}_0 \mathbf{u}_0 \qquad (U_{10}(t_0, \mathbf{x}_0) = 1), \tag{59}$$

$$P_j(t_0, \mathbf{x}_0) = 1 - U_{1j}(t_0, \mathbf{x}_0), \tag{60}$$

$$p_j(t_0, \mathbf{x}_0) = [U_{1j-1}(t_0, \mathbf{x}_0) - U_{1j}(t_0, \mathbf{x}_0)]/\Delta t, \tag{61}$$

$$T_j(t_0, \mathbf{x}_0) = \sum_{l=1}^{j-1} (t_l - t_0) p_l(t_0, \mathbf{x}_0) \Delta t, \tag{62}$$

$$\hat{T}_j(t_0, \mathbf{x}_0) = T_j(t_0, \mathbf{x}_0) + (t_j - t_0) U_{1j}(t_0, \mathbf{x}_0). \tag{63}$$

Theorem 2

Under the assumptions of Lemma 3 we have that

$$T^*(t_0, \mathbf{x}_0) = \lim_{j \to \infty} T_j(t_0, \mathbf{x}_0) = \lim_{j \to \infty} \hat{T}_j(t_0, \mathbf{x}_0) < \infty. \tag{64}$$

Proof. From (61) and (58), taking into account (50)–(52), we obtain

$$p_j(t_0, \mathbf{x}_0) \Delta t \leqslant c'(1 + 2\nu_0 \Delta t)^{-\varepsilon(j-1)/2};$$

thus from (62) it follows that

$$T_j(t_0, \mathbf{x}_0) \leqslant \Delta t c'(1 + 2\nu_0 \Delta t)^{\varepsilon/2} \sum_{l=1}^{j-1} l(1 + 2\nu_0 \Delta t)^{-\varepsilon l/2}. \tag{65}$$

Since

$$\lim_{l \to \infty} [l(1 + 2\nu_0 \Delta t)^{-\varepsilon l/2}]^{1/l} = (1 + 2\nu_0 \Delta t)^{-\varepsilon/2} < 1,$$

the series on the right of the inequality (65) is convergent. Moreover, since $\lim_{j \to \infty} (t_j - t_0) U_{1j} \leqslant \lim_{j \to \infty} j \Delta t c'(1 + 2\nu_0 \Delta t)^{-\varepsilon j/2} = 0$, (64) is proved. □

Remarks

(i) The matrices $\mathbf{D}(t)$ and $\mathbf{A}(t)$ depend on t through the depth $h(t, \mathbf{x})$, the matrix $\mathbf{B}(t)$ through the flux $h(t, \mathbf{x})\mathbf{v}(t, \mathbf{x})$. Since it is assumed that $|\eta(t, \mathbf{x})|_0 \ll h_0(\mathbf{x})$, in the computations we can approximate $h(t, \mathbf{x})$ by means of $h_0(\mathbf{x})$. Thus the matrices $\mathbf{D}(t)$ and $\mathbf{A}(t)$ become constant matrices.

(ii) Let

$$T_{2j}(t_0, \mathbf{x}_0) = \sum_{l=1}^{j-1} [t_l - t_0 - T^*(t_0, \mathbf{x}_0)]^2 p_l(t_0, \mathbf{x}_0)\Delta t$$

be an estimator of the variance of $\tau(t_0, \mathbf{x}_0)$. Under the assumptions of Theorem 2 it is possible to show that

$$\lim_{j\to\infty} T_{2j}(t_0, \mathbf{x}_0) < \infty.$$

4.2. Monte Carlo method

From the Monte Carlo point of view, a useful kind of discrete analogue of equation (2), with $\mathbf{w} = 0$, for the density function $u(t, \mathbf{x})$ in r-dimensional space ($r = 1, 2, 3$) is given by

$$u(t + \Delta t, \mathbf{x}^k) = \sum_{i=1}^{r} [Q_i(1 - Pe_i^k)]u(t, x_i^k + \Delta x_i) + \sum_{i=1}^{r} [Q_i(1 + Pe_i^k)]u(t, x_i^k - \Delta x_i) + \left[1 - 2\sum_{i=1}^{r} Q_i\right]u(t, \mathbf{x}^k),$$

where $\mathbf{x}^k$ is the kth grid point, Pe_i^k and Q_i are defined as

$$Pe_i^k = v_i^k \Delta x_i / 2k_i, \qquad Q_i = k_i \Delta t / \Delta x_i^2,$$

k_i is the diffusion coefficient for the ith direction, $\mathbf{v}^k$ is the velocity vector in $\mathbf{x}_k$ and the other symbols have an obvious meaning. The conditions

$$|Pe_i^k| < 1, \qquad 2\sum_{i=1}^{r} Q_i \leqslant 1$$

are assumed to be satisfied.

The terms in square brackets can in fact be interpreted as probabilities related to a random walk in the domain Ω. Starting from a source point in the domain, a fictitious particle jumps from one point to an adjacent one or, with probability given by the last term in square brackets, rests on the site. A walk is stopped according to the boundary conditions or when the total time of interest has elapsed.

In contrast, for a purely advective process ($k_i = 0$) an 'upwind' difference approximation has been assumed for the spatial differential operator, i.e.

$$u(t + \Delta t, \mathbf{x}^k) = \left[1 - \sum_{i=1}^{r} R_i^k\right]u(t, \mathbf{x}^k) + \sum_{i=1}^{r} R_i^k u(t, x_i + \Delta s),$$

where

$$R_i^k = |v_i^k|\Delta t/\Delta x_i, \qquad \Delta s = -\Delta x_i \quad \text{when } v_i^k \geqslant 0, \qquad \Delta s = \Delta x_i \quad \text{when } v_i^k < 0.$$

The condition

$$\sum_{i=1}^{r} R_i^k \leqslant 1$$

must be satisfied.

In the Monte Carlo simulations carried out in the present work, a particle is followed until it reaches the exit points on the boundary. If the velocity field on these points is oriented towards the exit direction, the particle is lost and the elapsed time accumulated into its proper counter. Otherwise the particle is driven back and the history continues. On boundaries different from the open ones a reflection condition is assumed, so that the hitting particle is compelled to return to the previous residence point. An estimate of the residence probability $U_1(t)$ at fixed time t is computed as

$$U_1(t) = 1 - Ne(t)/N,$$

where $Ne(t)$ is the total number of particles escaped before time t and N is the total number of source particles processed. At the same scoring time the mean residence time $\hat{T}(t)$ can be estimated as

$$\hat{T}(t) = \sum_{i=1}^{Ne(t)} (t_i - t_0)/N + (t - t_0)U_1(t),$$

where $t_i(<t)$ is the exit time of the ith escaped particle.

For both probability $U_1(t)$ and residence time $\hat{T}(t)$ a standard deviation is computed in the usual way, i.e.

$$\sigma(s(t)) = \sqrt{\left(\frac{\left(\sum_{i=1}^{N} s_i^2\right)/N - \bar{s}^2}{N}\right)},$$

where s_i is the current estimator for the quantity of interest $s(t)$ and $\bar{s}$ its average taken over the N source particles. In the numerical simulations performed with $N = 500$, we obtained $100\sigma(U_1(t))/U_1(t) < 10$ and $100\sigma(\hat{T}(t))/\hat{T}(t) < 1$.

5. NUMERICAL EXPERIMENTS

5.1. Sample problem 1: the two-dimensional lagoon

We consider a rectangular basin $L_1 \times L_2$ with three narrow openings of width d, indicated by A, B and C in Figure 2. A steady flow is obtained by imposing inflows Q_A in A and Q_B in B and an outflow $Q_C = Q_A + Q_B$ in C. The data of the sample problem are given in Table I. The streamlines of the flow are shown in Figure 2.

Numerical simulations have been performed by using finite volume and Monte Carlo methods for the pure advection problem and for the advection–diffusion problem P2. The mesh spacings and the time steps used in the two methods are different and are given in Table II. Here we present some results relative to the two initial positions (x_0, y_0) defined in Table I. Results obtained by Monte Carlo methods for a pure advection and an advection–diffusion problem are presented in Figure 3. For the pure advection problem we observe that $U_1(t; 0, \mathbf{x}_0)$ decreases abruptly at $t = T^*(0, \mathbf{x}_0) = \lim_{t\to\infty} \hat{T}(t; 0, \mathbf{x}_0) = 50$ days; the trend of $\hat{T}(t; 0, \mathbf{x}_0)$ is linear with unit slope for $t < T^*(0, \mathbf{x}_0)$ and $\hat{T}(t; 0, \mathbf{x}_0) = T^*(0, \mathbf{x}_0)$ for $t \geqslant T^*(0, \mathbf{x}_0)$, according to (19) and (20). The computed distributions $U_1(t; 0, \mathbf{x}_0)$ and estimators $\hat{T}(t; 0, \mathbf{x}_0)$ for problem P2 obtained by the two methods (Figure 4) are the

Table I. Data for sample problem 1

							(x_0, y_0) (m)	
L_1 (m)	L_2 (m)	d (m)	Q_A ($m^3 s^{-1}$)	Q_B ($m^3 s^{-1}$)	Q_C ($m^3 s^{-1}$)	k_H ($m^2 s^{-1}$)	I	II
3000	4000	50	4	2	6	2	(2940,800)	(1500,2000)

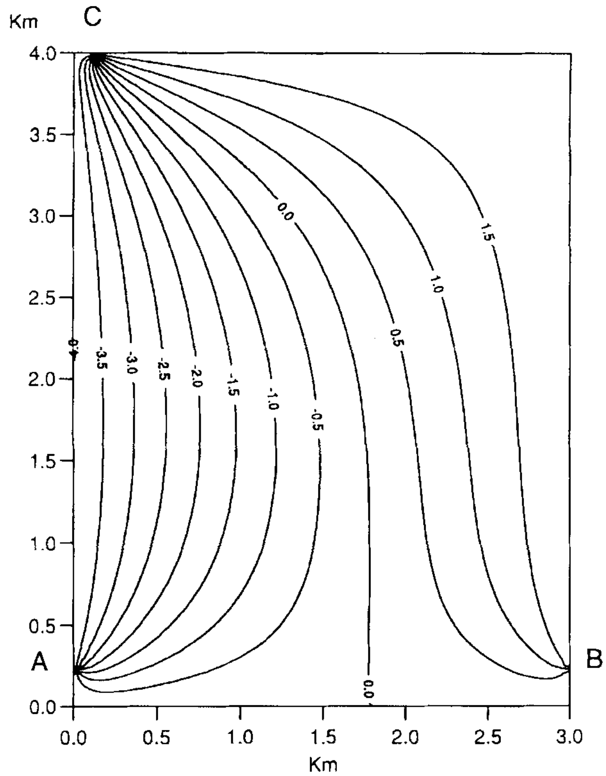

Figure 2. Rectangular basin with openings A, B (inflow) and C (outflow). The curves represent the streamlines of the flow

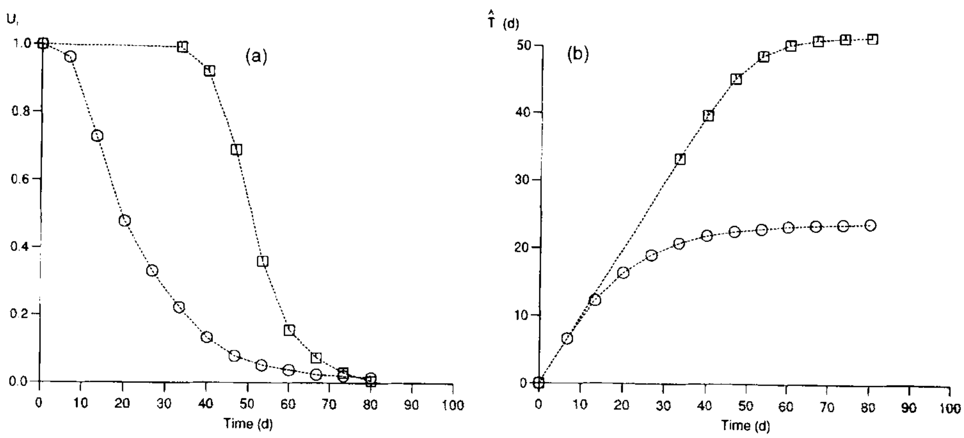

Figure 3. Sample problem 1. Comparison between pure advection (squares) and advection–diffusion (circles) problems; results by Monte Carlo method: (a) distribution $U_1(t;\, 0, \mathbf{x}_0)$; (b) estimator $\hat{T}(t;\, 0, \mathbf{x}_0)$ of residence time

same (differences of a few per cent). The densities $p(t;\, t_0,\, \mathbf{x}_0)$ are unimodal functions (Figure 5). The lines of equal concentration $u(t, \mathbf{x}) = \text{const.}$ and the particle distributions from Monte Carlo computations show that the space gradient of the concentration is sensitively dumped in a few days (Figure 6). A quantitative comparison between the two methods has been made by computing (Table III) the probabilities

$$\int_{R(t)} u(t, x, y)\, \mathrm{d}x\, \mathrm{d}y, \qquad \sum_{i \in R(t)} N_i(t)/N,$$

where $R(t)$ is the region closed by the line $u(t, x, y) = 10^{-7}$ and $N_i(t)$ is the number of particles at point i.

5.2. Sample problem 2: the one-dimensional channel

We consider the problem of a standing wave forced by the tide in a rectangular prismatic channel. We use the linearized one-dimensional approximatin governed by the wave equation. The channel of length L and constant depth h_0, is closed at one end $x = 0$ and at the open end $x = L$ the water level $\eta(t, x)$ is forced up and down according to

$$\eta(t, L) = a \sin(\omega t),$$

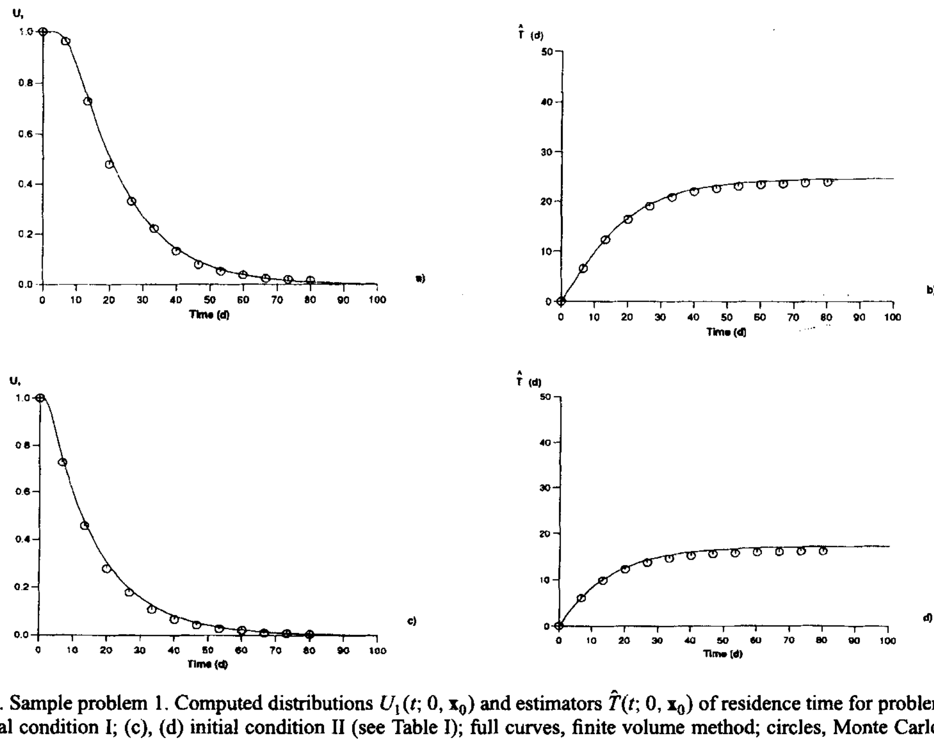

Figure 4. Sample problem 1. Computed distributions $U_1(t;\, 0,\, \mathbf{x}_0)$ and estimators $\hat{T}(t;\, 0,\, \mathbf{x}_0)$ of residence time for problem P2: (a), (b) initial condition I; (c), (d) initial condition II (see Table I); full curves, finite volume method; circles, Monte Carlo method

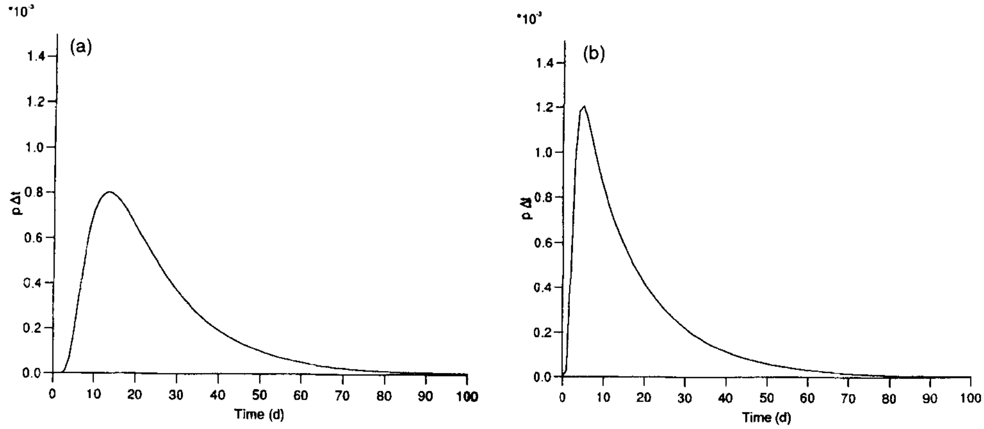

Figure 5. Sample problem 1. Computed densities $p(t; 0, \mathbf{x}_0)$ for problem P2 by finite volume method: (a) initial condition I; (b) initial condition II (see Table I)

where a is the amplitude and $\omega = 2\pi/\Theta$, with Θ the period of tidal oscillation. Let $v(t, x)$ be the component of velocity in the x-direction. At the solid boundary $x = 0$ we have $\partial\eta/\partial x = 0$ and $v(t, 0) = 0$. The solutions $\eta(t, x)$ and $v(t, x)$ of the wave equation are given by

$$\eta(t, x) = (a/\cos\beta)\cos(\beta x/L)\sin(\omega t), \tag{66}$$

$$v(t, x) = -\alpha\sin(\beta x/L)\cos(\omega t), \tag{67}$$

where

$$\alpha = a\sqrt{(gh_0)}/h_0\cos\beta, \qquad \beta = \omega L/\sqrt{(gh_0)}.$$

Table II. Mesh spacings and time steps for sample problem 1

	Finite volume (implicit time scheme)	Monte Carlo
Δx (m)	50	30
Δy (m)	50	40
Δt (s)	1800	100

Table III. Probabilities of finding a particle in the region $R(t)$ for the two numerical methods for sample problem 1

t (days)	$\int_{R(t)} u(t, x, y)\,dx\,dy$	$\sum_{i \in R(t)} N_i(t)/N$
1	0·9202	0·9180
2	0·8512	0·8340
3	0·7795	0·7500
4	0·7046	0·6740

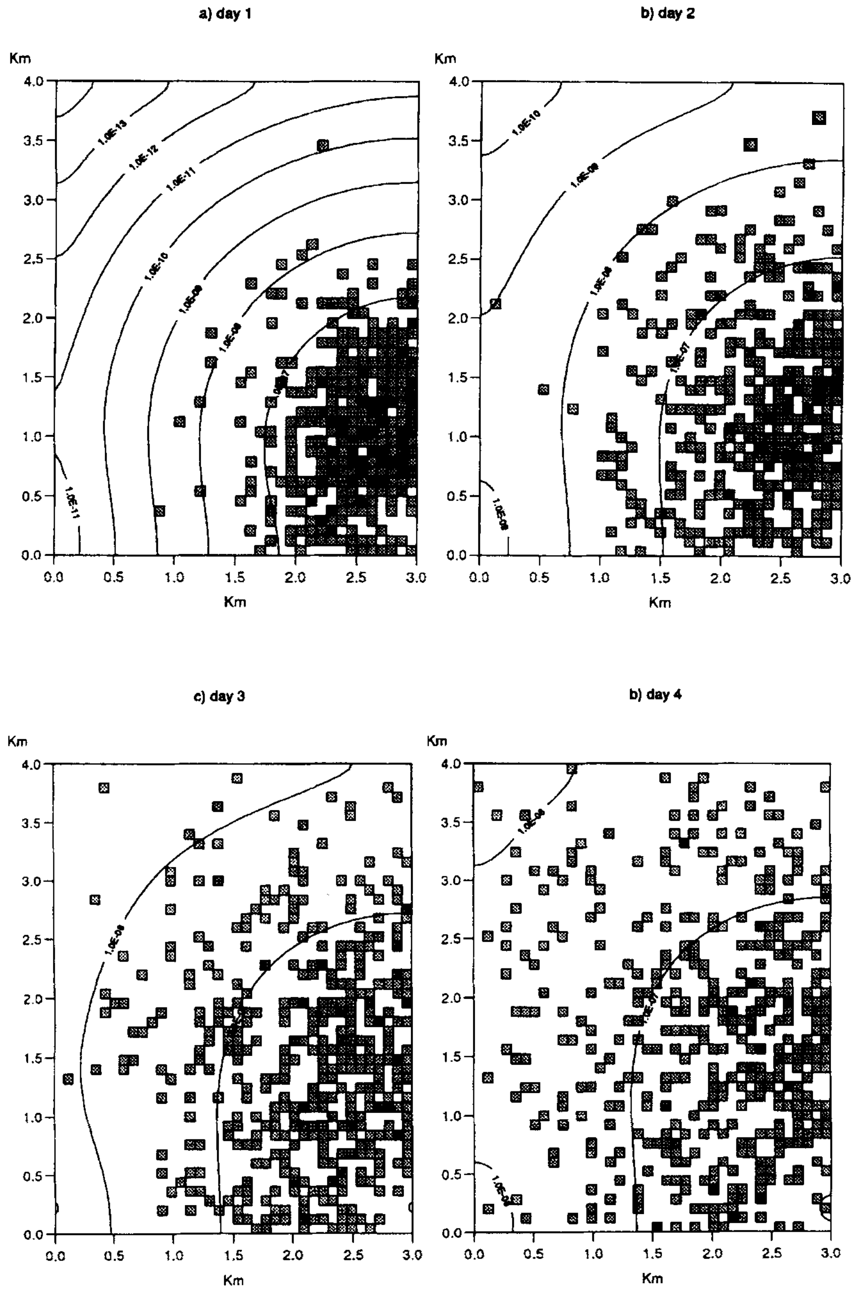

Figure 6. Sample problem 1. Lines of equal concentration $u(t, \mathbf{x}) = 10^{-k}$, $k = 6, 7, \ldots, 14$, for problem P2 and initial condition I, together with particle distributions (squares) from Monte Carlo computations

Table IV. Data for sample problem 2

L (m)	h_0 (m)	a (m)	ω (s^{-1})	α ($m\ s^{-1}$)	β ($m^2\ s^{-1}$)	k_H (m)	$\Delta x = L/n$ (s)	Δt (m)	x_0
2000	1·2	0·1	0·00014	0·25	0·082	2	20	300	600

Furthermore, we let $h(t, x) = h_0$ in (27). Under these assumptions we assume that $u(t, x)$ is a solution of a one-dimensional problem of type P1′ or P2′. This one-dimensional advection–diffusion model can be considered as either a depth-average model or a model for drogue floats (such as oranges, corks) on the sea surface. The data used in the numerical calculations are given in Table IV.

Let us consider the pure advective problem. The motion of a particle is governed by the equation.

$$\frac{dx(t)}{dt} = v(t, x(t)), \qquad x(0) = x_0,$$

where $v(t, x)$ is given by (67). It is easy to verify that the elapsed time $T^*(0, x_0)$ from the initial position $x(0) = x_0$ to the open boundary $x(T^*(0, x_0)) = L$ is given by

$$T^*(0, x_0) = (1/\omega) \sin^{-1}[(\omega/\alpha\beta) \log|\tan(\beta/2)/\tan(\beta x_0/2L)|]. \tag{68}$$

Let $\hat{x}_0$ be the value of x_0 for which the argument of the $\sin^{-1}$ function in the expression for $T^*(0, x_0)$ is equal to unity (note that $\beta < \pi$). Thus, for $L - x_0 \leqslant L - \hat{x}_0 = 140{\cdot}4\ m$, $T^*(0, x_0)$ is an increasing function of the distance $L - x_0$ between the initial position and the open boundary (Figure 7(a); for $L - x_0 > L - \hat{x}_0$ the particle never reaches the open boundary and $T^*(0, x_0) = \infty$. For comparison, $T^*(0, x_0)$ computed for a pure diffusion problem is shown in Figure 7(b).

Let us now consider the advection–diffusion problem P1′. In this case the condition on the open boundary is independent of the direction of the flow, so the particle always has a non-zero escape probability. Because of the oscillating velocity field, the distribution $U_1(t; 0, x_0)$ decreases more or less depending on the direction of the flow in the channel (Figure 8(a)). Moreover, the density $p(t; 0, x_0)$ is an oscillating function of the same period of the tide; the amplitude reaches a maximum and then vanishes

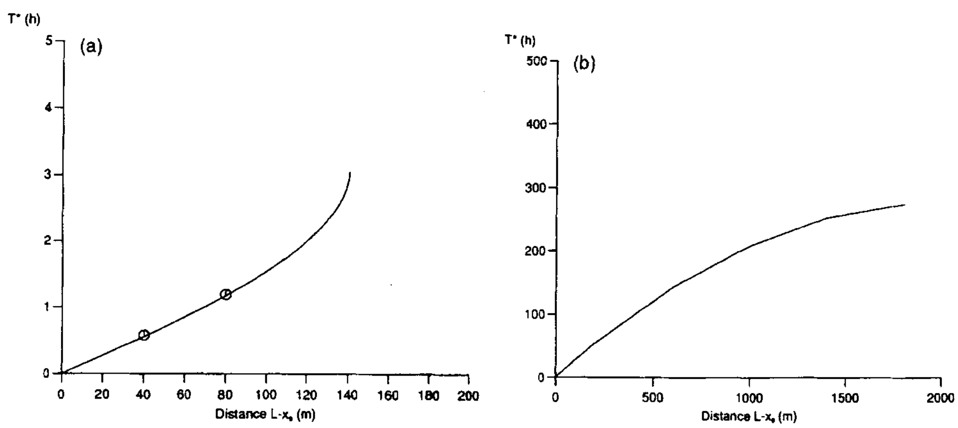

Figure 7. Sample problem 2. Residence time $T^*(0, x_0)$ versus $L - x_0$: (a) for advective motion in channel; full curve, equation (68); circles, Monte Carlo method; (b) for pure diffusive problem

Table V. Sample problem 2. Residence times and their standard deviations for pure diffusion problem and problems P1′ and P2′

Velocity field	Boundary conditions	T^* (days)	Standard deviation of T^* (days)
$\mathbf{v}=0$	$u=0$	10·5	9·3
Tidal oscillations	$u=0$	9·8	8·5
Tidal oscillations	See (31)	12·3	10·8

as $t \to \infty$ (Figure 8(b)). The residence time and its standard deviation (Figures 8(c) and 8(d)) are of the same order of magnitude. The estimator of the standard deviation (Figure 8(d)) shows a plateau for $t = T^*(0, x_0)$ (see the definition of $T_2(t; 0, x_0)$ in remark (iii) below Theorem 1).

For the advection–diffusion problem P2′ the condition on the open boundary is dependent on the direction of the flow, so the particle has a zero escape probability during the periods of inflow. Thus in these time intervals the slope of $U_1(t; 0, x_0)$ and the density $p(t; 0, x_0)$ are zero (Figures 9(a) and 9(b)). The trends of $T(t; 0, x_0)$ and $T_2(t; 0, x_0)$ (Figures 9(c) and 9(d)) show the same features as those for problem P1′. Tidal oscillations affect the trends of the residence time estimators much less than those of the probability densities (Figures 8(b), 8(c), 9(b) and 9(c)). The asymptotic values $T^*(0, x_0)$ for the considered situations are given in Table V.

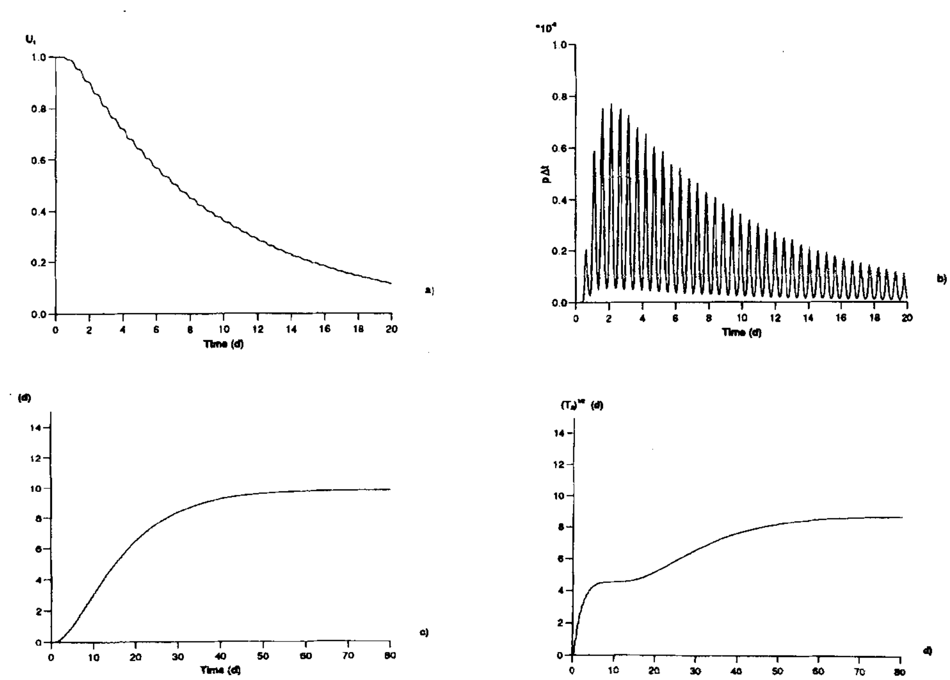

Figure 8. Sample problem 2. Finite volume results for problem P1′ in channel with flow induced by tide: (a) distribution $U_1(t; 0, \mathbf{x}_0)$; (b) density $p(t; 0, x_0)\Delta t$; (c) estimator $T(t; 0, x_0)$ of residence time; (d) estimator $[T_2(t; 0, x_0)]^{1/2}$ of standard deviation of residence time

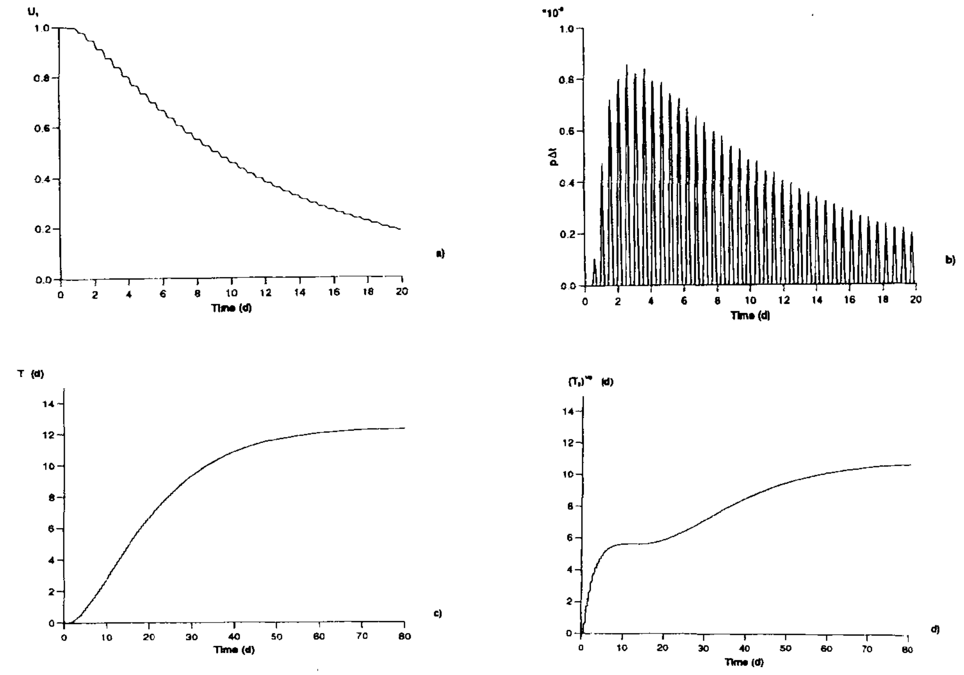

Figure 9. Sample problem 2. Finite volume results for problem P2′ in channel with flow induced by tide: (a) distribution $U_1(t; 0, \mathbf{x}_0)$; (b) density $p(t; 0, x_0)\Delta t$; (c) estimator $T(t; 0, x_0)$ of residence time, (d) estimator $[T_2(t; 0, x_0)]^{1/2}$ of standard deviation of residence time

6. CONCLUDING REMARKS

In this paper the problem of estimating the residence time of a passive constituent released at a given point in a semienclosed marine basin is studied. We define the random variable $\tau(t_0, \mathbf{x}_0)$ that represents the time spent by the particle in the basin. Its mean value $T^*(t_0, \mathbf{x}_0)$ is defined as the residence time of a particle released at time t_0 in $\mathbf{x}_0$. The basic assumptions of the model are: the stochastic dispersion process is described by the advection–diffusion equation; the flow field is either unidirectional in time or forced by the tide; the conditions on the fluid boundaries are such that all the particles eventually leave the basin.

Under these assumptions we prove that $T^*(t_0, \mathbf{x}_0)$ has a finite value. Two numerical models are used to solve the advection–diffusion equation: the finite volume and Monte Carlo methods. The results of the two models are quantitatively very similar for both the integrated quantities and the distribution of particles. A discrete approximation of problems with the flow forced by the tide, in the general shallow water formulation, is presented, its properties are shown and discrete analogues of Lemma 2 and Theorem 1 (see Lemma 4 and Theorem 2) are proved.

An investigation of dispersion processes and residence times in a large-scale basin where the flow field is characterized by recirculating gyres is in progress.[10] The numerical simulations of the dispersion processes are performed by means of Eulerian and Lagrangian models. The Lagrangian model consists of the numerical integration of a stochastic equation belonging to the general class of 'random flight models'.[11]

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